1 Stochastic Differential Equations

In this chapter, we consider general multidimensional SDEs of the form (1.1) given below.

In Section 1.1, we give the standard definitions of various types of the existence and the uniqueness of solutions as well as some general theorems that show the relationship between various properties.

Section 1.2 contains some classical sufficient conditions for various types of existence and uniqueness.

In Section 1.3, we present several important examples that illustrate various combinations of the existence and the uniqueness of solutions. Most of these examples (but not all) are well known. We also find all the possible combinations of existence and uniqueness.

Section 1.4 includes the definition of a martingale problem. We also recall the relationship between the martingale problems and the SDEs.

In Section 1.5, we define a solution up to a random time.

1.1 General Definitions

Here we will consider a general type of SDEs, i.e., multidimensional SDEs with coefficients that depend on the past. These are the equations of the form

$$dX_t^i = b_t^i(X)dt + \sum_{j=1}^m \sigma_t^{ij}(X)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n), \tag{1.1}$$

where $n \in \mathbb{N}$, $m \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$, and

$$b : C(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n,$$
$$\sigma : C(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}$$

are predictable functionals. (The definition of a predictable process can be found, for example, in [27, Ch. I, §2 a] or [38, Ch. IV, § 5].)

Remark. We fix a starting point x_0 together with b and σ. In our terminology, SDEs with the same b and σ and with different starting points are different SDEs.

A.S. Cherny and H.-J. Engelbert: LNM 1858, pp. 5–25, 2005.

Definition 1.1. **(i)** A *solution* of (1.1) is a pair (Z, B) of adapted processes on a filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, \mathsf{Q}\big)$ such that

(a) B is a m-dimensional $(\mathcal{G}_t)$-Brownian motion, i.e., B is a m-dimensional Brownian motion started at zero and is a $(\mathcal{G}_t, \mathsf{Q})$-martingale;

(b) for any $t \ge 0$,

$$\int_0^t \left(\sum_{i=1}^n |b_s^i(Z)| + \sum_{i=1}^n \sum_{j=1}^m (\sigma_s^{ij}(Z))^2 \right) ds < \infty \quad \mathsf{Q}\text{-a.s.};$$

(c) for any $t \ge 0$, $i = 1, \dots, n$,

$$Z_t^i = x_0^i + \int_0^t b_s^i(Z)ds + \sum_{j=1}^m \int_0^t \sigma_s^{ij}(Z)dB_s^j \quad \mathsf{Q}\text{-a.s.}$$

(ii) There is *weak existence* for (1.1) if there exists a solution of (1.1) on some filtered probability space.

Definition 1.2. **(i)** A solution (Z, B) is called a *strong solution* if Z is $\big(\overline{\mathcal{F}}_t^B\big)$-adapted, where $\overline{\mathcal{F}}_t^B$ is the σ-field generated by $\sigma(B_s;\ s \le t)$ and by the subsets of the Q-null sets from $\sigma(B_s;\ s \ge 0)$.

(ii) There is *strong existence* for (1.1) if there exists a strong solution of (1.1) on some filtered probability space.

Remark. Solutions in the sense of Definition 1.1 are sometimes called *weak solutions*. Here we call them simply *solutions*. However, the existence of a solution is denoted by the term *weak existence* in order to stress the difference between weak existence and *strong existence* (i.e., the existence of a strong solution).

Definition 1.3. There is *uniqueness in law* for (1.1) if for any solutions (Z, B) and $(\widetilde{Z}, \widetilde{B})$ (that may be defined on different filtered probability spaces), one has $\text{Law}(Z_t;\ t \ge 0) = \text{Law}(\widetilde{Z}_t;\ t \ge 0)$.

Definition 1.4. There is *pathwise uniqueness* for (1.1) if for any solutions (Z, B) and $(\widetilde{Z}, B)$ (that are defined on the same filtered probability space), one has $\mathsf{Q}\{\forall t \ge 0,\ Z_t = \widetilde{Z}_t\} = 1$.

Remark. If there exists no solution of (1.1), then there are both uniqueness in law and pathwise uniqueness.

The following 4 statements clarify the relationship between various properties.

Proposition 1.5. *Let* (Z, B) *be a strong solution of* (1.1).

(i) *There exists a measurable map*

$$\Psi : \big(C(\mathbb{R}_+, \mathbb{R}^m), \mathcal{B}\big) \longrightarrow \big(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{B}\big)$$

(*here* $\mathcal{B}$ *denotes the Borel* σ*-field*) *such that the process* $\Psi(B)$ *is* $\big(\overline{\mathcal{F}}_t^B\big)$*-adapted and* $Z = \Psi(B)$ Q*-a.s.*

(ii) *If* $\widetilde{B}$ *is a m-dimensional* $(\widetilde{\mathcal{F}}_t)$*-Brownian motion on a filtered probability space* $\big(\widetilde{\Omega}, \widetilde{\mathcal{G}}, (\widetilde{\mathcal{G}}_t), \widetilde{\mathsf{Q}}\big)$ *and* $\widetilde{Z} := \Psi(\widetilde{B})$*, then* $(\widetilde{Z}, \widetilde{B})$ *is a strong solution of* (1.1)*.*

For the proof, see, for example, [5].

Now we state a well known result of Yamada and Watanabe.

Proposition 1.6 (Yamada, Watanabe). *Suppose that pathwise uniqueness holds for* (1.1)*.*

(i) *Uniqueness in law holds for* (1.1);

(ii) *There exists a measurable map*

$$\Psi : \big(C(\mathbb{R}_+, \mathbb{R}^m), \mathcal{B}\big) \longrightarrow \big(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{B}\big)$$

such that the process $\Psi(B)$ *is* $\big(\overline{\mathcal{F}}_t^B\big)$*-adapted and, for any solution* (Z, B) *of* (1.1)*, we have* $Z = \Psi(B)$ Q*-a.s.*

For the proof, see [48] or [38, Ch. IX, Th. 1.7].

The following result complements the theorem of Yamada and Watanabe.

Proposition 1.7. *Suppose that uniqueness in law holds for* (1.1) *and there exists a strong solution. Then pathwise uniqueness holds for* (1.1)*.*

This theorem was proved by Engelbert [10] under some additional assumptions. It was proved with no additional assumptions by Cherny [7].

The crucial fact needed to prove Proposition 1.7 is the following result. It shows that uniqueness in law implies a seemingly stronger property.

Proposition 1.8. *Suppose that uniqueness in law holds for* (1.1)*. Then, for any solutions* (Z, B) *and* $(\widetilde{Z}, \widetilde{B})$ (*that may be defined on different filtered probability spaces*)*, one has* $\mathrm{Law}(Z_t, B_t;\, t \ge 0) = \mathrm{Law}(\widetilde{Z}_t, \widetilde{B}_t;\, t \ge 0)$*.*

For the proof, see [7].

The situation with solutions of SDEs can now be described as follows.

It may happen that there exists no solution of (1.1) on any filtered probability space (see Examples 1.16, 1.17).

It may also happen that on some filtered probability space there exists a solution (or there are even several solutions with the same Brownian motion), while on some other filtered probability space with a Brownian motion there exists no solution (see Examples 1.18, 1.19, 1.20, and 1.24).

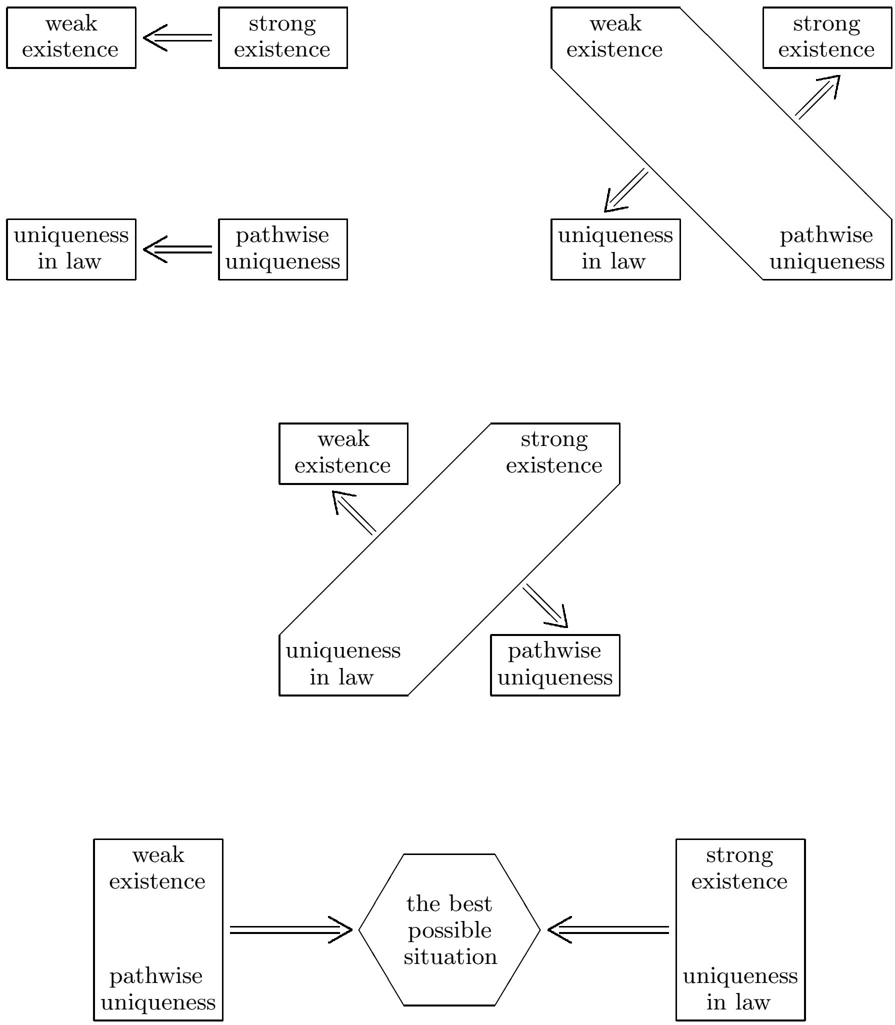

Fig. 1.1. The relationship between various types of existence and uniqueness. The top diagrams show obvious implications and the implications given by the Yamada–Watanabe theorem. The centre diagram shows an obvious implication and the implication given by Proposition 1.7. The bottom diagram illustrates the Yamada–Watanabe theorem and Proposition 1.7 in terms of the “best possible situation”.

If there exists a strong solution of (1.1) on some filtered probability space, then there exists a strong solution on any other filtered probability space with a Brownian motion (see Proposition 1.5). However, it may happen in this case that there are several solutions with the same Brownian motion (see Examples 1.21–1.23).

If pathwise uniqueness holds for (1.1) and there exists a solution on some filtered probability space, then on any other filtered probability space with a Brownian motion there exists exactly one solution, and this solution is strong (see the Yamada–Watanabe theorem). This is the best possible situation.

Thus, the Yamada–Watanabe theorem shows that pathwise uniqueness together with weak existence guarantee that the situation is the best possible. Proposition 1.7 shows that uniqueness in law together with strong existence guarantee that the situation is the best possible.

1.2 Sufficient Conditions for Existence and Uniqueness

The statements given in this section are related to SDEs, for which $b_t(X) = b(t, X_t)$ and $\sigma_t(X) = \sigma(t, X_t)$, where $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions.

We begin with sufficient conditions for strong existence and pathwise uniqueness. The first result of this type was obtained by Itô.

Proposition 1.9 (Itô). *Suppose that, for a SDE*

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{m} \sigma^{ij}(t, X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

there exists a constant $C > 0$ such that

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le C\|x - y\|, \quad t \ge 0,\ x, y \in \mathbb{R}^n,$$
$$\|b(t,x)\| + \|\sigma(t,x)\| \le C(1 + \|x\|), \quad t \ge 0,\ x \in \mathbb{R}^n,$$

where

$$\|b(t,x)\| := \left(\sum_{i=1}^{n} (b^i(t,x))^2\right)^{1/2},$$
$$\|\sigma(t,x)\| := \left(\sum_{i=1}^{n}\sum_{j=1}^{m} (\sigma^{ij}(t,x))^2\right)^{1/2}.$$

Then strong existence and pathwise uniqueness hold.

For the proof, see [25], [29, Ch. 5, Th. 2.9], or [36, Th. 5.2.1].

Proposition 1.10 (Zvonkin). *Suppose that, for a one-dimensional SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0,$$

the coefficient b is measurable and bounded, the coefficient σ is continuous and bounded, and there exist constants $C > 0$, $\varepsilon > 0$ such that

$$|\sigma(t,x) - \sigma(t,y)| \le C\sqrt{|x-y|}, \quad t \ge 0,\ x, y \in \mathbb{R},$$
$$|\sigma(t,x)| \ge \varepsilon, \quad t \ge 0,\ x \in \mathbb{R}.$$

Then strong existence and pathwise uniqueness hold.

For the proof, see [49].

For homogeneous SDEs, there exists a stronger result.

Proposition 1.11 (Engelbert, Schmidt). *Suppose that, for a one-dimensional SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0,$$

$\sigma \ne 0$ at each point, $b/\sigma^2 \in L^1_{\mathrm{loc}}(\mathbb{R})$, and there exists a constant $C > 0$ such that

$$|\sigma(x) - \sigma(y)| \le C\sqrt{|x-y|}, \quad x, y \in \mathbb{R},$$
$$|b(x)| + |\sigma(x)| \le C(1 + |x|), \quad x \in \mathbb{R}.$$

Then strong existence and pathwise uniqueness hold.

For the proof, see [15, Th. 5.53].

The following proposition guarantees only pathwise uniqueness. Its main difference from Proposition 1.10 is that the diffusion coefficient here need not be bounded away from zero.

Proposition 1.12 (Yamada, Watanabe). *Suppose that, for a one-dimensional SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0,$$

there exist a constant $C > 0$ and a strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^{0+} h^{-2}(x)dx = +\infty$ such that

$$|b(t,x) - b(t,y)| \le C|x-y|, \quad t \ge 0,\ x, y \in \mathbb{R},$$
$$|\sigma(t,x) - \sigma(t,y)| \le h(|x-y|), \quad t \ge 0,\ x, y \in \mathbb{R}.$$

Then pathwise uniqueness holds.

For the proof, see [29, Ch. 5, Prop. 2.13], [38, Ch. IX, Th. 3.5], or [39, Ch. V, Th. 40.1].

We now turn to results related to weak existence and uniqueness in law. The first of these results guarantees only weak existence; it is almost covered by further results, but not completely. Namely, here the diffusion matrix σ need not be elliptic (it might even be not a square matrix).

Proposition 1.13 (Skorokhod). *Suppose that, for a SDE*

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{m} \sigma^{ij}(t, X_t)dB_t^j, \quad X_0^i = x_0^i \quad (i = 1, \dots, n),$$

the coefficients b and σ are continuous and bounded. Then weak existence holds.

For the proof, see [42] or [39, Ch. V, Th. 23.5].

Remark. The conditions of Proposition 1.13 guarantee neither strong existence (see Example 1.19) nor uniqueness in law (see Example 1.22).

In the next result, the conditions on b and σ are essentially relaxed as compared with the previous proposition.

Proposition 1.14 (Stroock, Varadhan). *Suppose that, for a SDE*

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(t, X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

the coefficient b is measurable and bounded, the coefficient σ is continuous and bounded, and, for any $t \geq 0$, $x \in \mathbb{R}^n$, there exists a constant $\varepsilon(t, x) > 0$ such that

$$\|\sigma(t, x)\lambda\| \geq \varepsilon(t, x)\|\lambda\|, \quad \lambda \in \mathbb{R}^n.$$

Then weak existence and uniqueness in law hold.

For the proof, see [44, Th. 4.2, 5.6].

In the next result, the diffusion coefficient σ need not be continuous. However, the statement deals with homogeneous SDEs only.

Proposition 1.15 (Krylov). *Suppose that, for a SDE*

$$dX_t^i = b^i(X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

the coefficient b is measurable and bounded, the coefficient σ is measurable and bounded, and there exist a constant $\varepsilon > 0$ such that

$$\|\sigma(x)\lambda\| \geq \varepsilon\|\lambda\|, \quad x \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n.$$

Then weak existence holds. If moreover $n \leq 2$, then uniqueness in law holds.

For the proof, see [32].

Remark. In the case $n > 2$, the conditions of Proposition 1.15 do not guarantee uniqueness in law (see Example 1.24).

1.3 Ten Important Examples

In the examples given below, we will use the *characteristic diagrams* $\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }$ to illustrate the statement of each example. The first square in the diagram corresponds to weak existence; the second – to strong existence; the third – to uniqueness in law; the fourth – to pathwise uniqueness. Thus, the statement "for the SDE ..., we have $\boxed{+}\boxed{-}\boxed{+}\boxed{-}$" should be read as follows: "for the SDE ..., there exists a solution, there exists no strong solution, uniqueness in law holds, and pathwise uniqueness does not hold".

We begin with examples of SDEs with no solution.

Example 1.16 (no solution). *For the SDE*

$$dX_t = -\operatorname{sgn} X_t dt, \quad X_0 = 0, \tag{1.2}$$

where

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \le 0, \end{cases} \tag{1.3}$$

we have $\boxed{-}\boxed{-}\boxed{+}\boxed{+}$.

Proof. Suppose that there exists a solution (Z, B). Then almost all paths of Z satisfy the integral equation

$$f(t) = -\int_0^t \operatorname{sgn} f(s) ds, \quad t \ge 0. \tag{1.4}$$

Let f be a solution of this equation. Assume that there exist $a > 0$, $t > 0$ such that $f(t) = a$. Set $v = \inf\{t \ge 0 : f(t) = a\}$, $u = \sup\{t \le v : f(t) = 0\}$. Using (1.4), we get $a = f(v) - f(u) = -(v - u)$. The obtained contradiction shows that $f \le 0$. In a similar way we prove that $f \ge 0$. Thus, $f \equiv 0$, but then it is not a solution of (1.4). As a result, (1.4), and hence, (1.2), has no solution. □

The next example is a SDE with the same characteristic diagram and with $\sigma \equiv 1$.

Example 1.17 (no solution). *For the SDE*

$$dX_t = -\frac{1}{2X_t} I(X_t \ne 0) dt + dB_t, \quad X_0 = 0, \tag{1.5}$$

we have $\boxed{-}\boxed{-}\boxed{+}\boxed{+}$.

Proof. Suppose that (Z, B) is a solution of (1.5). Then

$$Z_t = -\int_0^t \frac{1}{2Z_s} I(Z_s \neq 0)ds + B_t, \quad t \geq 0.$$

By Itô's formula,

$$\begin{aligned} Z_t^2 &= -\int_0^t 2Z_s \frac{1}{2Z_s} I(Z_s \neq 0)ds + \int_0^t 2Z_s dB_s + t \\ &= \int_0^t I(Z_s = 0)ds + \int_0^t 2Z_s dB_s, \quad t \geq 0. \end{aligned}$$

The process Z is a continuous semimartingale with $\langle Z \rangle_t = t$. Hence, by the occupation times formula,

$$\int_0^t I(Z_s = 0)ds = \int_{\mathbb{R}} I(x = 0) L_t^x(Z)dx = 0, \quad t \geq 0,$$

where $L_t^x(Z)$ denotes the local time of the process Z (see Definition A.2). As a result, Z^2 is a positive local martingale, and consequently, a supermartingale. Since $Z^2 \geq 0$ and $Z_0^2 = 0$, we conclude that $Z^2 = 0$ a.s. But then (Z, B) is not a solution of (1.5). □

Now we turn to the examples of SDEs that possess a solution, but no strong solution.

Example 1.18 (no strong solution; Tanaka). *For the SDE*

$$dX_t = \operatorname{sgn} X_t dB_t, \quad X_0 = 0 \tag{1.6}$$

(*for the precise definition of* sgn, *see* (1.3)), *we have* $\boxed{+|-|+|-}$.

Proof. Let W be a Brownian motion on $(\Omega, \mathcal{G}, \mathsf{Q})$. We set

$$Z_t = W_t, \quad B_t = \int_0^t \operatorname{sgn} W_s dW_s, \quad t \geq 0$$

and take $\mathcal{G}_t = \mathcal{F}_t^W$. Obviously, (Z, B) is a solution of (1.6) on $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$.

If (Z, B) is a solution of (1.6) on a filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$, then Z is a continuous $(\mathcal{G}_t, \mathsf{Q})$-local martingale with $\langle Z \rangle_t = t$. It follows from P. Lévy's characterization theorem that Z is a Brownian motion. This implies uniqueness in law.

If (Z, B) is a solution of (1.6), then

$$B_t = \int_0^t \operatorname{sgn} Z_s dZ_s, \quad t \geq 0.$$

This implies that $\mathcal{F}_t^B = \mathcal{F}_t^{|Z|}$ (see [38, Ch. VI, Cor. 2.2]). Hence, there exists no strong solution.

If (Z, B) is a solution of (1.6), then $(-Z, B)$ is also a solution. Thus, there is no pathwise uniqueness. □

The next example is a SDE with the same characteristic diagram, $b = 0$, and a continuous σ.

Example 1.19 (no strong solution; Barlow). *There exists a continuous bounded function* $\sigma : \mathbb{R} \to (0, \infty)$ *such that, for the SDE*

$$dX_t = \sigma(X_t) dB_t, \quad X_0 = x_0,$$

we have $\boxed{+ \mid - \mid + \mid -}$ *.*

For the proof, see [2].

The next example is a SDE with the same characteristic diagram and with $\sigma \equiv 1$. The drift coefficient in this example depends on the past.

Example 1.20 (no strong solution; Tsirelson). *There exists a bounded predictable functional* $b : C(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$ *such that, for the SDE*

$$dX_t = b_t(X) dt + dB_t, \quad X_0 = x_0,$$

we have $\boxed{+ \mid - \mid + \mid -}$ *.*

For the proof, see [46], [23, Ch. IV, Ex. 4.1], or [38, Ch. IX, Prop. 3.6].

Remark. Let B be a Brownian motion on $(\Omega, \mathcal{G}, \mathsf{Q})$. Set $\mathcal{G}_t = \mathcal{F}_t^B$. Then the SDEs of Examples 1.18–1.20 have no solution on $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$ with the Brownian motion B. Indeed, if (Z, B) is a solution, then Z is $(\mathcal{G}_t)$-adapted, which means that (Z, B) is a strong solution.

We now turn to examples of SDEs, for which there is no uniqueness in law.

Example 1.21 (no uniqueness in law). *For the SDE*

$$dX_t = I(X_t \neq 0) dB_t, \quad X_0 = 0, \tag{1.7}$$

we have $\boxed{+ \mid + \mid - \mid -}$ *.*

Proof. It is sufficient to note that (B, B) and $(0, B)$ are solutions of (1.7) on $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$ whenever B is a $(\mathcal{G}_t)$-Brownian motion. □

Remark. Let B be a Brownian motion on $(\Omega, \mathcal{G}, \mathsf{Q})$ and η be a random variable that is independent of B with $\mathsf{P}\{\eta = 1\} = \mathsf{P}\{\eta = -1\} = 1/2$. Consider

$$Z_t(\omega) = \begin{cases} B_t(\omega) & \text{if } \eta(\omega) = 1, \\ 0 & \text{if } \eta(\omega) = -1 \end{cases}$$

and take $\mathcal{G}_t = \mathcal{F}_t^Z$. Then (Z, B) is a solution of (1.7) on $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$ that is not strong. Indeed, for each $t > 0$, η is not $\overline{\mathcal{F}}_t^B$-measurable. Since the sets $\{\eta = -1\}$ and $\{Z_t = 0\}$ are indistinguishable, Z_t is not $\overline{\mathcal{F}}_t^B$-measurable.

The next example is a SDE with the same characteristic diagram, $b = 0$, and a continuous σ.

Example 1.22 (no uniqueness in law; Girsanov). *Let* $0 < \alpha < 1/2$. *Then, for the SDE*

$$dX_t = |X_t|^\alpha dB_t, \quad X_0 = 0, \tag{1.8}$$

we have $\boxed{+|+|-|-}$.

Proof. Let W be a Brownian motion started at zero on $(\Omega, \mathcal{G}, \mathsf{Q})$ and

$$\begin{aligned} A_t &= \int_0^t |W_s|^{-2\alpha} ds, \quad t \ge 0, \\ \tau_t &= \inf\{s \ge 0 : A_s > t\}, \quad t \ge 0, \\ Z_t &= W_{\tau_t}, \quad t \ge 0. \end{aligned}$$

The occupation times formula and Proposition A.6 (ii) ensure that A_t is a.s. continuous and finite. It follows from Proposition A.9 that $A_t \xrightarrow[t\to\infty]{\text{a.s.}} \infty$. Hence, τ is a.s. finite, continuous, and strictly increasing. By Proposition A.16, Z is a continuous $(\mathcal{F}^W_{\tau_t})$-local martingale with $\langle Z \rangle_t = \tau_t$. Using Proposition A.18, we can write

$$\tau_t = \int_0^{\tau_t} ds = \int_0^{\tau_t} |W_s|^{2\alpha} dA_s = \int_0^{A_{\tau_t}} |W_{\tau_s}|^{2\alpha} ds = \int_0^t |Z_s|^{2\alpha} ds, \quad t \ge 0.$$

(We have $A_{\tau_t} = t$ due to the continuity of A and the property $A_t \xrightarrow[t\to\infty]{\text{a.s.}} \infty$.) Hence, the process

$$B_t = \int_0^t |Z_s|^{-\alpha} dZ_s, \quad t \ge 0$$

is a continuous $(\mathcal{F}^W_{\tau_t})$-local martingale with $\langle B \rangle_t = t$. According to P. Lévy's characterization theorem, B is a $(\mathcal{F}^W_{\tau_t})$-Brownian motion. Thus, (Z, B) is a solution of (1.8).

Now, all the desired statements follow from the fact that $(0, B)$ is another solution of (1.8). □

The next example is a SDE with the same characteristic diagram and with $\sigma \equiv 1$.

Example 1.23 (no uniqueness in law; SDE for a Bessel process). *For the SDE*

$$dX_t = \frac{\delta - 1}{2X_t} I(X_t \ne 0) dt + dB_t, \quad X_0 = 0 \tag{1.9}$$

with $\delta > 1$, *we have* $\boxed{+|+|-|-}$.

Proof. It follows from Proposition A.21 that there exists a solution (Z,B) of (1.9) such that Z is positive. By Itô's formula,

$$\begin{aligned} Z_t^2 &= \int_0^t (\delta-1)I(Z_s\neq 0)ds + 2\int_0^t Z_s dB_s + t \\ &= \delta t - \int_0^t (\delta-1)I(Z_s=0)ds + 2\int_0^t \sqrt{|Z_s^2|}dB_s, \quad t\geq 0. \end{aligned}$$

By the occupation times formula,

$$\int_0^t I(Z_s=0)ds = \int_0^t I(Z_s=0)d\langle Z\rangle_s = \int_{\mathbb{R}} I(x=0)L_t^x(Z)dx = 0, \quad t\geq 0.$$

Hence, the pair (Z^2,B) is a solution of the SDE

$$dX_t = \delta dt + 2\sqrt{|X_t|}dB_t, \quad X_0=0.$$

Propositions 1.6 and 1.12 combined together show that Z^2 is $\big(\overline{\mathcal{F}}_t^B\big)$-adapted. As Z is positive, Z is also $\big(\overline{\mathcal{F}}_t^B\big)$-adapted, which means that (Z,B) is a strong solution.

By Proposition 1.5 (i), there exists a measurable map $\Psi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ such that the process $\Psi(B)$ is $\big(\overline{\mathcal{F}}_t^B\big)$-adapted and $Z=\Psi(B)$ a.s. For any $t\geq 0$, we have

$$\Psi_t(B) = \int_0^t \frac{\delta-1}{2\Psi_s(B)} I\big(\Psi_s(B)\neq 0\big)ds + B_t \quad \text{a.s.}$$

The process $\widetilde{B}=-B$ is a Brownian motion. Hence, for any $t\geq 0$,

$$-\Psi_t(-B) = \int_0^t \frac{\delta-1}{-2\Psi_s(-B)} I\big(-\Psi_s(-B)\neq 0\big)ds + B_t \quad \text{a.s.}$$

Consequently, the pair $(\widetilde{Z},B)$, where $\widetilde{Z}=-\Psi(-B)$, is a (strong) solution of (1.9). Obviously, Z is positive, while $\widetilde{Z}$ is negative. Hence, Z and $\widetilde{Z}$ have a.s. different paths and different laws. This implies that there is no uniqueness in law and no pathwise uniqueness for (1.9). □

Remark. More information on SDE (1.9) can be found in [5]. In particular, it is proved in [5] that this equation possesses solutions that are not strong. Moreover, it is shown that, for the SDE

$$dX_t = \frac{\delta-1}{2X_t} I(X_t\neq 0)dt + dB_t, \quad X_0=x_0 \tag{1.10}$$

(here the starting point x_0 is arbitrary) with $1<\delta<2$, we have $\boxed{+}\boxed{+}\boxed{-}\boxed{-}$; for SDE (1.10) with $\delta\geq 2$, $x_0\neq 0$, we have $\boxed{+}\boxed{+}\boxed{+}\boxed{+}$. The SDE for a Bessel process is also considered in Sections 2.2, 3.4.

The following rather surprising example has multidimensional nature.

Example 1.24 (no uniqueness in law; Nadirashvili). *Let* $n \geq 3$. *There exists a function* $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ *such that*

$$\varepsilon \|\lambda\| \leq \|\sigma(x)\lambda\| \leq C\|\lambda\|, \quad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}^n$$

with some constants $C > 0$, $\varepsilon > 0$ *and, for the SDE*

$$dX_t^i = \sum_{j=1}^{n} \sigma^{ij}(X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

we have $\begin{array}{|c|c|c|c|}\hline + & & - & - \\ \hline\end{array}$.

For the proof, see [35] or [40].

We finally present one more example. Its characteristic diagram is different from all the diagrams that appeared so far.

Example 1.25 (no strong solution and no uniqueness). *For the SDE*

$$dX_t = \sigma(t, X_t)dB_t, \quad X_0 = 0 \tag{1.11}$$

with

$$\sigma(t,x) = \begin{cases} \operatorname{sgn} x & \text{if } t \leq 1, \\ I(x \neq 1)\operatorname{sgn} x & \text{if } t > 1 \end{cases}$$

(*for the precise definition of* sgn, *see* (1.3)), *we have* $\begin{array}{|c|c|c|c|}\hline + & - & - & - \\ \hline\end{array}$.

Proof. If W is a Brownian motion, then the pair

$$Z_t = W_t, \quad B_t = \int_0^t \operatorname{sgn} W_s dW_s, \quad t \geq 0 \tag{1.12}$$

is a solution of (1.11).

Let (Z, B) be the solution given by (1.12). Set $\tau = \inf\{t \geq 1 : Z_t = 1\}$, $\widetilde{Z}_t = Z_{t \wedge \tau}$. Then $(\widetilde{Z}, B)$ is another solution. Thus, there is no uniqueness in law and no pathwise uniqueness.

If (Z, B) is a solution of (1.12), then

$$Z_t = \int_0^t \operatorname{sgn} Z_s dB_s, \quad t \leq 1.$$

The arguments used in the proof of Example 1.18 show that (Z, B) is not a strong solution. □

Table 1.1. Possible and impossible combinations of existence and uniqueness. As an example, the combination "$+-+-$" in line 11 corresponds to a SDE, for which there exists a solution, there exists no strong solution, there is uniqueness in law, and there is no pathwise uniqueness. The table shows that such a SDE is provided by each of Examples 1.18–1.20.

Weak existence	**Strong existence**	**Uniqueness in law**	**Pathwise uniqueness**	**Possible/Impossible**
−	−	−	−	impossible, obviously
−	−	−	+	impossible, obviously
−	−	+	−	impossible, obviously
−	−	+	+	possible, Examples 1.16,1.17
−	+	−	−	impossible, obviously
−	+	−	+	impossible, obviously
−	+	+	−	impossible, obviously
−	+	+	+	impossible, obviously
+	−	−	−	possible, Example 1.25
+	−	−	+	impossible, Proposition 1.6
+	−	+	−	possible, Examples 1.18–1.20
+	−	+	+	impossible, Proposition 1.6
+	+	−	−	possible, Examples 1.21–1.23
+	+	−	+	impossible, Proposition 1.6
+	+	+	−	impossible, Proposition 1.7
+	+	+	+	possible, obviously

Remark. The SDE

$$dX_t = I(X_t \neq 1)\operatorname{sgn} X_t dB_t, \quad X_0 = 0$$

is a homogeneous SDE with the same characteristic diagram as in Example 1.25. However, it is more difficult to prove that this equation has no strong solution.

Let us mention one of the applications of the results given above. For SDE (1.1), each of the following properties:

weak existence,
strong existence,
uniqueness in law,
pathwise uniqueness

may hold or may not hold. Thus, there are 16 ($= 2^4$) feasible combinations. Some of these combinations are impossible (for instance, if there is pathwise uniqueness, then there must be uniqueness in law). For each of these combinations, Propositions 1.6, 1.7 and Examples 1.16–1.25 allow one either to provide an example of a corresponding SDE or to prove that this combination is impossible. It turns out that there are only 5 possible combinations (see Table 1.1).

1.4 Martingale Problems

Let $n \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and

$$b : C(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n,$$
$$a : C(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}_+ \longrightarrow \mathbb{R}^{n \times n}$$

be predictable functionals. Suppose moreover that, for any $t \geq 0$ and $\omega \in C(\mathbb{R}_+, \mathbb{R}^n)$, the matrix $a_t(\omega)$ is positively definite.

Throughout this section, $X = (X_t;\, t \geq 0)$ will denote the *coordinate process* on $C(\mathbb{R}_+, \mathbb{R}^n)$, i.e., the process defined by

$$X_t : C(\mathbb{R}_+, \mathbb{R}^n) \ni \omega \longmapsto \omega(t) \in \mathbb{R}^n.$$

By $(\mathcal{F}_t)$ we will denote the *canonical filtration* on $C(\mathbb{R}_+)$, i.e., $\mathcal{F}_t = \sigma(X_s;\, s \leq t)$, and $\mathcal{F}$ will stand for the σ-field $\bigvee_{t \geq 0} \mathcal{F}_t = \sigma(X_s;\, s \geq 0)$. Note that $\mathcal{F}$ coincides with the Borel σ-field $\mathcal{B}(C(\mathbb{R}_+, \mathbb{R}^n))$.

Definition 1.26. A *solution of the martingale problem* (x_0, b, a) is a measure P on $\mathcal{B}(C(\mathbb{R}_+, \mathbb{R}^n))$ such that

(a) $\mathsf{P}\{X_0 = x_0\} = 1$;

(b) for any $t \geq 0$,

$$\int_0^t \Bigl(\sum_{i=1}^n |b_s^i(X)| + \sum_{i=1}^n a^{ii}(X)\Bigr) ds < \infty \quad \mathsf{P}\text{-a.s.};$$

(c) for any $i = 1, \dots, n$, the process

$$M_t^i = X_t^i - \int_0^t b_s^i(X) ds, \quad t \geq 0 \tag{1.13}$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale;

(d) for any $i, j = 1, \dots, n$, the process

$$M_t^i M_t^j - \int_0^t a_s^{ij}(X) ds, \quad t \geq 0$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale.

Let us now consider SDE (1.1) and set

$$a_t(\omega) = \sigma_t(\omega)\sigma_t^*(\omega), \quad t \ge 0,\ \omega \in C(\mathbb{R}_+, \mathbb{R}^n),$$

where σ^* denotes the transpose of the matrix σ. Then the martingale problem (x_0, b, a) is called a *martingale problem corresponding* to SDE (1.1). The relationship between (1.1) and this martingale problem becomes clear from the following statement.

Theorem 1.27. (i) *Let* (Z, B) *be a solution of* (1.1). *Then the measure* $\mathsf{P} = \mathrm{Law}(Z_t;\, t \ge 0)$ *is a solution of the martingale problem* (x_0, b, a).

(ii) *Let* P *be a solution of the martingale problem* (x_0, b, a). *Then there exist a filtered probability space* $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$ *and a pair of processes* (Z, B) *on this space such that* (Z, B) *is a solution of* (1.1) *and* $\mathrm{Law}(Z_t;\, t \ge 0) = \mathsf{P}$.

Proof. **(i)** Conditions (a), (b) of Definition 1.26 are obviously satisfied. Let us check condition (c). Set

$$N_t = Z_t - \int_0^t b_s(Z)ds, \quad t \ge 0.$$

(We use here the vector form of notation.) For $m \in \mathbb{N}$, we consider the stopping time $S_m(N) = \inf\{t \ge 0 : \|N_t\| \ge m\}$. Since N is a $(\mathcal{G}_t, \mathsf{Q})$-local martingale, the stopped process $N^{S_m(N)}$ is a $(\mathcal{G}_t, \mathsf{Q})$-martingale. Hence, for any $0 \le s < t$ and $C \in \mathcal{F}_s$, we have

$$\mathsf{E}_{\mathsf{Q}}\big[\big(N_t^{S_m(N)} - N_s^{S_m(N)}\big)I(Z \in C)\big] = 0.$$

Therefore,

$$\mathsf{E}_{\mathsf{P}}\big[\big(M_t^{S_m(M)} - M_s^{S_m(M)}\big)I(X \in C)\big] = 0,$$

where M is given by (1.13) and $S_m(M) = \inf\{t \ge 0 : \|M_t\| \ge m\}$. This proves that $M \in \mathcal{M}^c_{\mathrm{loc}}(\mathcal{F}_t, \mathsf{P})$. Condition (d) of Definition 1.26 is verified in a similar way.

(ii) (Cf. [39, Ch. V, Th. 20.1].) Let $\Omega^1 = C(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{G}^1_t = \mathcal{F}_t$, $\mathcal{G}^1 = \mathcal{F}$, $\mathsf{Q}^1 = \mathsf{P}$. Choose a filtered probability space $\big(\Omega^2, \mathcal{G}^2, (\mathcal{G}^2_t), \mathsf{Q}^2\big)$ with a m-dimensional $(\mathcal{G}^2_t)$-Brownian motion W and set

$$\Omega = \Omega^1 \times \Omega^2, \quad \mathcal{G} = \mathcal{G}^1 \times \mathcal{G}^2, \quad \mathcal{G}_t = \mathcal{G}^1_t \times \mathcal{G}^2_t, \quad \mathsf{Q} = \mathsf{Q}^1 \times \mathsf{Q}^2.$$

We extend the processes b, σ, a from Ω^1 to Ω and the process W from Ω^2 to Ω in the obvious way.

For any $t \ge 0$, $\omega \in \Omega$, the matrix $\sigma_t(\omega)$ corresponds to a linear operator $\mathbb{R}^m \to \mathbb{R}^n$. Let $\varphi_t(\omega)$ be the $m \times m$-matrix of the operator of orthogonal projection onto $(\ker \sigma_t(\omega))^\perp$, where $\ker \sigma_t(\omega)$ denotes the kernel of $\sigma_t(\omega)$; let $\psi_t(\omega)$ be the $m \times m$-matrix of the operator of orthogonal projection onto $\ker \sigma_t(\omega)$. Then $\varphi = (\varphi_t;\, t \ge 0)$ and $\psi = (\psi_t;\, t \ge 0)$ are predictable $\mathbb{R}^{m \times m}$-valued processes. For any $t \ge 0$, $\omega \in \Omega$, the restriction of the operator $\sigma_t(\omega)$

to $(\ker\sigma_t(\omega))^\perp$ is a bijection from $(\ker\sigma_t(\omega))^\perp \subseteq \mathbb{R}^m$ onto $\operatorname{Im}\sigma_t(\omega) \subseteq \mathbb{R}^n$, where $\operatorname{Im}\sigma_t(\omega)$ denotes the image of $\sigma_t(\omega)$. Let us define the operator $\chi_t(\omega) : \mathbb{R}^n \to \mathbb{R}^m$ as follows: $\chi_t(\omega)$ maps $\operatorname{Im}\sigma_t(\omega)$ onto $(\ker\sigma_t(\omega))^\perp$ as the inverse of $\sigma_t(\omega)$; $\chi_t(\omega)$ vanishes on $(\operatorname{Im}\sigma_t(\omega))^\perp$. Obviously, $\chi = (\chi_t;\, t \geq 0)$ is a predictable $\mathbb{R}^{m\times n}$-valued process. We have $\chi_t(\omega)\sigma_t(\omega) = \varphi_t(\omega)$.

Define the process Z as $Z_t(\omega^1, \omega^2) = \omega^1(t)$ and the process M as

$$M_t = Z_t - \int_0^t b_s ds, \quad t \geq 0.$$

Let us set

$$B_t = \int_0^t \chi_s dM_s + \int_0^t \psi_s dW_s, \quad t \geq 0.$$

(We use here the vector form of notation.) For any $i, j = 1, \dots, n$, we have

$$\begin{aligned}
\langle B^i, B^j\rangle_t &= \int_0^t \sum_{k,l=1}^n \chi_s^{ik} a_s^{kl} \chi_s^{jl} ds + \int_0^t \sum_{k=1}^n \psi_s^{ik}\psi_s^{jk} ds \\
&= \int_0^t (\chi_s\sigma_s\sigma_s^*\chi_s^*)^{ij} ds + \int_0^t (\psi_s\psi_s^*)^{ij} ds \\
&= \int_0^t (\varphi_s\varphi_s^*)^{ij} ds + \int_0^t (\psi_s\psi_s^*)^{ij} ds \\
&= \int_0^t \big((\varphi_s + \psi_s)(\varphi_s^* + \psi_s^*)\big)^{ij} ds \\
&= \int_0^t \delta^{ij} ds = \delta^{ij} t, \quad t \geq 0.
\end{aligned}$$

By the multidimensional version of P. Lévy's characterization theorem (see [38, Ch. IV, Th. 3.6]), we deduce that B is a m-dimensional $(\mathcal{G}_t)$-Brownian motion.

Set $\rho_t(\omega) = \sigma_t(\omega)\chi_t(\omega)$. Let us consider the process

$$N_t = \int_0^t \sigma_s dB_s = \int_0^t \sigma_s\chi_s dM_s + \int_0^t \sigma_s\psi_s dW_s = \int_0^t \rho_s dM_s, \quad t \geq 0.$$

Then, for any $i = 1, \dots, n$, we have

$$\begin{aligned}
\langle N^i\rangle_t &= \int_0^t (\rho_s a_s \rho_s^*)^{ii} ds = \int_0^t (\sigma_s\chi_s\sigma_s\sigma_s^*\chi_s^*\sigma_s^*)^{ii} ds \\
&= \int_0^t (\sigma_s\sigma_s^*)^{ii} ds = \int_0^t a_s^{ii} ds = \langle M^i\rangle_t, \quad t \geq 0.
\end{aligned} \tag{1.14}$$

(We have used the obvious equality $\sigma_s\chi_s\sigma_s = \sigma_s$.) Furthermore,

$$\begin{aligned}\langle N^i, M^i\rangle_t &= \int_0^t (\rho_s a_s)^{ii} ds = \int_0^t (\sigma_s \chi_s \sigma_s \sigma_s^*)^{ii} ds \\ &= \int_0^t (\sigma_s \sigma_s^*)^{ii} ds = \int_0^t a_s^{ii} ds = \langle M^i\rangle_t, \quad t \geq 0.\end{aligned} \tag{1.15}$$

Comparing (1.14) with (1.15), we deduce that $\langle N^i - M^i\rangle = 0$. Hence, $M = x_0 + N$. As a result, the pair (Z, B) is a solution of (1.1). □

In this monograph, we will investigate only weak solutions and uniqueness in law for SDE (1). It will be more convenient for us to consider a solution of (1) as a solution of the corresponding martingale problem rather than to treat it in the sense of Definition 1.1. The reason is that in this case a solution is a single object and not a pair of processes as in Definition 1.1. This approach is justified by Theorem 1.27. Thus, from here on, we will always deal with the following definition, which is a reformulation of Definition 1.26 for the case of the SDEs having the form (1).

Definition 1.28. A *solution* of SDE (1) is a measure P on $\mathcal{B}(C(\mathbb{R}_+))$ such that

(a) $\mathsf{P}\{X_0 = x_0\} = 1$;

(b) for any $t \geq 0$,

$$\int_0^t \big(|b(X_s)| + \sigma^2(X_s)\big) ds < \infty \quad \mathsf{P}\text{-a.s.};$$

(c) the process

$$M_t = X_t - \int_0^t b(X_s) ds, \quad t \geq 0 \tag{1.16}$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale;

(d) the process

$$M_t^2 - \int_0^t \sigma^2(X_s) ds, \quad t \geq 0$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale.

Remark. If one accepts Definition 1.28, then the *existence* and *uniqueness* of a solution are defined in an obvious way. It follows from Theorem 1.27 that the existence of a solution in the sense of Definition 1.28 is equivalent to weak existence (Definition 1.1); the uniqueness of a solution in the sense of Definition 1.28 is equivalent to uniqueness in law (Definition 1.3).

Definition 1.29. **(i)** A solution P of (1) is *positive* if $\mathsf{P}\{\forall t \geq 0,\, X_t \geq 0\} = 1$.

(ii) A solution P of (1) is *strictly positive* if $\mathsf{P}\{\forall t \geq 0,\, X_t > 0\} = 1$.

The *negative* and *strictly negative* solutions are defined in a similar way.

1.5 Solutions up to a Random Time

There are several reasons why we consider solutions up to a random time. First, a solution may explode. Second, a solution may not be extended after it reaches some level. Third, we can guarantee in some cases that a solution exists up to the first time it leaves some interval, but we cannot guarantee the existence of a solution after that time (see Chapter 2).

In order to define a solution up to a random time, we replace the space $C(\mathbb{R}_+)$ of continuous functions by the space $\overline{C}(\mathbb{R}_+)$ defined below. We need this space to consider exploding solutions. Let π be an isolated point added to the real line.

Definition 1.30. The space $\overline{C}(\mathbb{R}_+)$ consists of the functions $f : \mathbb{R}_+ \to \mathbb{R} \cup \{\pi\}$ with the following property: there exists a time $\xi(f) \in [0, \infty]$ such that f is continuous on $[0, \xi(f))$ and $f = \pi$ on $[\xi(f), \infty)$. The time $\xi(f)$ is called the *killing time* of f.

Throughout this section, $X = (X_t;\, t \geq 0)$ will denote the *coordinate process* on $\overline{C}(\mathbb{R}_+)$, i.e.,

$$X_t : \overline{C}(\mathbb{R}_+) \ni \omega \longmapsto \omega(t) \in \mathbb{R} \cup \{\pi\},$$

$(\mathcal{F}_t)$ will denote the *canonical filtration* on $\overline{C}(\mathbb{R}_+)$, i.e., $\mathcal{F}_t = \sigma(X_s;\, s \leq t)$, and $\mathcal{F}$ will stand for the σ-field $\bigvee_{t \geq 0} \mathcal{F}_t = \sigma(X_s;\, s \geq 0)$.

Remark. There exists a metric on $\overline{C}(\mathbb{R}_+)$ with the following properties.

(a) It turns $\overline{C}(\mathbb{R}_+)$ into a Polish space.

(b) The convergence $f_n \to f$ in this metric is equivalent to:

$$\xi(f_n) \xrightarrow[n \to \infty]{} \xi(f);$$
$$\forall t < \xi(f), \quad \sup_{s \leq t} |f_n(s) - f(s)| \xrightarrow[n \to \infty]{} 0.$$

(In particular, $C(\mathbb{R}_+)$ is a closed subspace in this metric.)

(c) The Borel σ-field on $\overline{C}(\mathbb{R}_+)$ with respect to this metric coincides with $\sigma(X_t;\, t \geq 0)$.

In what follows, we will need two different notions: a solution up to S and a solution up to $S-$.

Definition 1.31. Let S be a stopping time on $\overline{C}(\mathbb{R}_+)$. A *solution of* (1) *up to* S (or a *solution defined up to* S) is a measure P on $\mathcal{F}_S$ such that

(a) $\mathsf{P}\{\forall t \leq S,\, X_t \neq \pi\} = 1$;

(b) $\mathsf{P}\{X_0 = x_0\} = 1$;

(c) for any $t \geq 0$,

$$\int_0^{t \wedge S} \big(|b(X_s)| + \sigma^2(X_s)\big) ds < \infty \quad \mathsf{P}\text{-a.s.};$$

(d) the process

$$M_t = X_{t\wedge S} - \int_0^{t\wedge S} b(X_s)ds, \quad t \ge 0 \tag{1.17}$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale;

(e) the process

$$M_t^2 - \int_0^{t\wedge S} \sigma^2(X_s)ds, \quad t \ge 0$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale.

In the following, we will often say that (P, S) *is a solution of* (1).

Remarks. (i) The measure P is defined on $\mathcal{F}_S$ and not on $\mathcal{F}$ since otherwise it would not be unique.

(ii) In the usual definition of a local martingale, the probability measure is defined on $\mathcal{F}$. Here P is defined on a smaller σ-field $\mathcal{F}_S$. However, in view of the equality $M^S = M$, the knowledge of P only on $\mathcal{F}_S$ is sufficient to verify the inclusion $M \in \mathcal{M}_{\text{loc}}^c(\mathcal{F}_t, \mathsf{P})$ that arises in (d). In other words, if $\widetilde{\mathsf{P}}$ and $\widetilde{\mathsf{P}}'$ are probability measures on $\mathcal{F}$ such that $\widetilde{\mathsf{P}}|\mathcal{F}_S = \widetilde{\mathsf{P}}'|\mathcal{F}_S = \mathsf{P}$, then $M \in \mathcal{M}_{\text{loc}}^c(\mathcal{F}_t, \widetilde{\mathsf{P}})$ if and only if $M \in \mathcal{M}_{\text{loc}}^c(\mathcal{F}_t, \widetilde{\mathsf{P}}')$ (so we can write simply $M \in \mathcal{M}_{\text{loc}}^c(\mathcal{F}_t, \mathsf{P})$). In order to prove this statement, note that the inclusion $M \in \mathcal{M}_{\text{loc}}^c(\mathcal{F}_t, \widetilde{\mathsf{P}})$ means that there exists a sequence of stopping times (S_n) such that

(a) $S_n \le S_{n+1}$;

(b) $S_n \le S$;

(c) for any $t \ge 0$, $\widetilde{\mathsf{P}}\{t\wedge S_n = t\wedge S\} \xrightarrow[n\to\infty]{} 1$ (note that $\{t \wedge S_n = t \wedge S\} \in \mathcal{F}_S$);

(d) for any $s \le t$, $C \in \mathcal{F}_s$, and $n \in \mathbb{N}$,

$$\mathsf{E}_{\widetilde{\mathsf{P}}}\big[\big(M_{t\wedge S_n} - M_{s\wedge S_n}\big)I(C)\big] = 0.$$

This expression makes sense since the random variable

$$\big(M_{t\wedge S_n} - M_{s\wedge S_n}\big)I(C) = \big(M_{t\wedge S_n} - M_{s\wedge S_n}\big)I(C \cap \{S_n > s\})$$

is $\mathcal{F}_S$-measurable.

Similarly, in order to verify conditions (a), (b), (c), and (e), it is sufficient to know the values of P only on $\mathcal{F}_S$.

Definition 1.32. **(i)** A solution (P, S) is *positive* if $\mathsf{P}\{\forall t \le S,\ X_t \ge 0\} = 1$.

(ii) A solution (P, S) is *strictly positive* if $\mathsf{P}\{\forall t \le S,\ X_t > 0\} = 1$.

The *negative* and *strictly negative* solutions are defined in a similar way.

Recall that a function $S : \overline{C}(\mathbb{R}_+) \to [0, \infty]$ is called a *predictable stopping time* if there exists a sequence (S_n) of $(\mathcal{F}_t)$-stopping times such that

(a) $S_n \le S_{n+1}$;

(b) $S_n \le S$ and $S_n < S$ on the set $\{S > 0\}$;

(c) $\lim_n S_n = S$.

In the following, we will call (S_n) a *predicting sequence* for S.

Definition 1.33. Let S be a predictable stopping time on $\overline{C}(\mathbb{R}_+)$ with a predicting sequence (S_n). A *solution of* (1) *up to* $S-$ (or a *solution defined up to* $S-$) is a measure P on $\mathcal{F}_{S-}$ such that, for any $n \in \mathbb{N}$, the restriction of P to $\mathcal{F}_{S_n}$ is a solution up to S_n.

In the following, we will often say that $(\mathsf{P}, S-)$ *is a solution of* (1).

Remarks. (i) Obviously, this definition does not depend on the choice of a predicting sequence for S.

(ii) Definition 1.33 implies that $\mathsf{P}\{\forall t < S,\, X_t \neq \pi\} = 1$.

(iii) When dealing with solutions up to S, one may use the space $C(\mathbb{R}_+)$. The space $\overline{C}(\mathbb{R}_+)$ is essential only for solutions up to $S-$.

In this monograph, we will use the following terminology: a solution in the sense of Definition 1.28 will be called a *global* solution, while a solution in the sense of Definition 1.31 or Definition 1.33 will be called a *local* solution. The next statement clarifies the relationship between these two notions.

Theorem 1.34. **(i)** *Suppose that* (P, S) *is a solution of* (1) *in the sense of Definition* 1.31 *and* $S = \infty$ P-*a.s. Then* P *admits a unique extension* $\widetilde{\mathsf{P}}$ *to* $\mathcal{F}$. *Let* Q *be the measure on* $C(\mathbb{R}_+)$ *defined as the restriction of* $\widetilde{\mathsf{P}}$ *to* $\{\xi = \infty\} = C(\mathbb{R}_+)$. *Then* Q *is a solution of* (1) *in the sense of Definition* 1.28.

(ii) *Let* Q *be a solution of* (1) *in the sense of Definition* 1.28. *Let* P *be the measure on* $\overline{C}(\mathbb{R}_+)$ *defined as* $\mathsf{P}(A) = \mathsf{Q}(A \cap \{\xi = \infty\})$. *Then* (P, ∞) *is a solution of* (1) *in the sense of Definition* 1.31.

Proof. **(i)** The existence and the uniqueness of $\widetilde{\mathsf{P}}$ follow from Lemma B.5.

The latter part of **(i)** as well as statement **(ii)** are obvious. □